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EAR-DECOMPOSITIONS OF MATCHING-COVERED GRAPHS

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Dedicated to Tibor Gallai on his seventieth birthday

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We call a graph *matching-covered* if every line belongs to a perfect matching. We study the technique of "ear-decompositions" of such graphs. We prove that a non-bipartite matching-covered graph contains K_4 or $K_2 \oplus K_3$ (the triangular prism). Using this result, we give new characterizations of those graphs whose matching and covering numbers are equal. We apply these results to the theory of τ-critical graphs.

0. Introduction

In this paper we study the structure of those connected graphs, called *matching-covered,* in which every line belongs to a perfect matching. This is, of course, a rather restrictive property but quite often (e.g., when studying the number of perfect matchings) it can be assumed without loss of generality by simply ignoring those lines which do not occur in any perfect matching.

Two main lines in the study of structural aspects of matching-covered graphs started quite early. Kotzig [10] defined the "canonical partition" of these graphs. Some further properties and applications of this partition were given by Lovász [13]. The other technique of decomposing such graphs is the technique of "ear-decompositions". This was first introduced by Hetyei [9], and refined by Lovász and Plummer [15].

In this paper we further exploit this ear-decomposition technique, and give some applications of the results. In Chapter 1 some results on ear-decompositions are proved. In Chapter 2 we prove that if a matching-covered graph is non-bipartite then it has to contain one of two non-bipartite matching-covered graphs.

Sterboul [20] and Deming [3] characterized those graphs in which the matching number equals the point-covering number (e.g. all bipartite graphs have this property by König's theorem). We offer two further characterizations of this property in Chapter 3, in part as an application of the main result in Chapter 2. The second one of these is closely related to Sterboul's and Deming's, even though no direct way is known to derive one from the other. We also show how to apply these results in the theory of τ-critical graphs, and formulate some conjectures.

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1. Preliminaries

Let G be a finite, undirected graph. We say that G is *critical* if $G-x$ has a perfect matching for every $x \in V(G)$. We say that G is *matching-covered* if it is connected and every line of G belongs to a perfect matching of G. Matching-covered graphs have been studied under various names (elementary graphs without forbidden lines, Lovász and Plummer [15]; 1-extendable graphs, Plummer [19]; U-graphs, Naddef [17] etc.). One early result on their structure is due to Kotzig [10] and Lovász [13]:

Theorem 1.1 *Let G be a matching-covered graph. Then $V(G)$ has a (uniquely determined) partition $\mathcal{P}(G)$ with the following properties:*
(a) *For $x, y \in V(G)$, $G-x-y$ has a perfect matching iff x and y belong to different classes of $\mathcal{P}(G)$;*
(b) *A subset $S \subseteq V(G)$ is a class of $\mathcal{P}(G)$ iff $G-S$ has $|S|$ connected components, each of which is critical.* □

In fact, this result was proved for a somewhat larger class of graphs called *elementary*, but we shall not need this fact.

It follows from (a) above that the classes of $\mathcal{P}(G)$ are independent sets, i.e. $\mathcal{P}(G)$ is a coloration of G. If G is bipartite than it is not too difficult to show that $\mathcal{P}(G)$ coincides with its unique 2-coloration. (The study of bipartite matching-covered graphs goes back all the way to König [11]). If the graph G is not bipartite then $\mathcal{P}(G)$ is in general far from the best possible coloration. E.g. if we subdivide every line of K_{2p} by two points we obtain a graph G with chromatic number 3 and with $2p$ classes in $\mathcal{P}(G)$.

It may be worthwhile to formulate in general how the partition $\mathcal{P}(G)$ changes under subdivision. A graph G' is called an *even subdivision* of G if every line of G is subdivided by an even number of new points (possibly 0).

Proposition 1.2. *Let G be a matching-covered graph and G' an even subdivision of G. Then $\mathcal{P}(G')$, restricted to $V(G)$, is the same partiton as $\mathcal{P}(G)$. Furthermore, if x is a point of $V(G')-V(G)$ which subdivides a line $uv \in E(G)$ and is at an even distance from u on this subdivided line, then x belongs to the same class of $\mathcal{P}(G')$ as u.* □

Another approach to the structure of matching-covered graphs is due to Hetyei [9]. Let G be a graph and G' a subgraph of G. We say that G' is a *nice subgraph* if $G-V(G')$ has a perfect matching. (Thus every spanning subgraph is nice.)

Let G be a graph and G' a subgraph of G. A path $P \subset G$ is called an *ear* of G' if $V(P) \cap V(G')$ consists of the two endpoints of P. In this paper all ears will have odd lengths and we shall not tell this explicitely. Any line spanned by $V(G')$ but not in $E(G')$ is an ear of G'.

A *(weak) ear-decomposition* of G starting with G' is a representation G in the form

$$G = G' \cup P_1 \cup \ldots \cup P_k,$$

where P_{i+1} is an (odd) ear of $G' \cup P_1 \cup \ldots \cup P_i$ for $i=0, 1, \ldots, k-1$. The starting point of our paper is the following result, implicit in the work of Hetyei [1964].

Theorem 1.3. *Let G be a matching-covered graph and G' a subgraph of G. Then G has an ear-decomposition starting with G' if and only if G' is a nice subgraph of G.*

Proof. The fact that if G has an ear-decomposition starting with G' then G' is nice is trivial: for every ear P_i, let $M(P_i)$ denote the set of lines of P_i at an odd distance from its endpoints. Then $M(P_1)\cup\ldots\cup M(P_k)$ is a perfect matching of $G-V(G')$.

On the other hand, assume that G' is nice and let M be a perfect matching of $G-V(G')$. If G' is spanning then the assetion is obvious, so suppose that G' is not spanning. Since G is connected, there exists a line e connecting $V(G')$ to $V(G)-V(G')$. Let F be a perfect matching of G containing e. Let P_1 be the connected component of $M\cup F$ containing e. Then P_1 is clearly a path alternating with respect to M and starting and ending at G'. Hence P_1 is an ear of G' and $G'\cup P_1$ is a nice subgraph of G. We can continue in the same manner and find an ear-decomposition of G. □

A *strong ear decomposition* is an ear-decomposition starting with a single line, in which the ears are grouped:

$$G = \{e_0\}\cup(P_1\cup\ldots\cup P_{i_1})\cup(P_{i_1+1}\cup\ldots\cup P_{i_2})\cup\ldots\cup(P_{i_{r-1}+1}\cup\ldots\cup P_{i_r}),$$

in such a way that the paths in each paranthesis are vertex-disjoint and, for every $1\leqq t\leqq r$, the graph $\{e_0\}\cup(P_1\cup\ldots\cup P_{i_1})\cup\ldots\cup(P_{i_{t-1}+1}\cup\ldots\cup P_{i_t})$ is matching-covered.

Example: Let $G=K_4$, where $V(K_4)=\{1,2,3,4\}$. Then $G=12\cup 2341\cup(14\cup 23)$ is a strong ear-decomposition of G (abusing notation, we denote the path through the points 2, 3, 4, 1 simply by 2341; also we omit a pair of parantheses if there is only one ear inside). Note that the second paranthesis is necessary since if only one of 14 or 23 is added to the cycle 12341, then the resulting graph is not matching-covered.

The following result is due to Lovász and Plummer [15]:

Theorem 1.4. *Every matching-covered graph has a strong ear-decomposition with at most two ears in each paranthesis.* □

The fact that every matching-covered graph has a strong ear-decomposition was proved by Hetyei; this follows by the same argument as Theorem 1.3. It is considerably more complicated to prove that the parantheses can be broken up into others containing at most two ears so that we still obtain a strong ear-decomposition. More exactly one can prove the following, slightly stronger result:

Theorem 1.5. *Let G be a matching-covered graph and G' a matching-covered nice subgraph of G. Then there exist one or two disjoint ears of G' in G such that adding these to G' results in a matching-covered subgraph.* □

One may formulate this result in a third way. Let us call a strong ear-decomposition of a matching-covered graph G *non-refinable* if we cannot break up any paranthesis into two; in other words, if we consider any paranthesis

$$G = \ldots(P_i\cup\ldots\cup P_j)\ldots$$

in this ear-decomposition, and denote the subgraph constructed previously by G_1, then for every non-empty proper subset $\{i_1,\ldots,i_m\}\subset\{i,\ldots,j\}$, the subgraph $G_i\cup P_{i_1}\cup\ldots\cup P_{i_m}$ is not matching-covered.

Theorem 1.6. *In any non-refinable ear-decomposition of G, every paranthesis consists of one or two ears.* □

We shall call a paranthesis accordingly a *1- or 2-ear addition step.*

The case of bipartite graphs is again of particular interest. The following result was proved (in a matrix form) by Hartfietd [8].

Theorem 1.7. *A bipartite graph is matching-covered if and only if it has a weak ear-decomposition starting with a line. A matching-covered graph is bipartite if and only if it has a strong ear-decomposition with no parantheses. Every weak ear-decomposition of a matching-covered bipartite graph is strong.* □

We next prove two lemmas by an "ear-decomposition" technique which will enable us to find special ears of subgraphs.

Lemma 1.8. (Ear Selection Lemma). *Let G be a matching-covered graph, G_0 a nice subgraph of G, and $V(G_0)=V_1\cup V_2$ a partition of $V(G_0)$. Assume that there exists a path in $G-E(G_0)$ connecting V_1 to V_2. Then there exists an ear of G_0 connecting V_1 to V_2.*

Proof. We use induction on $|E(G)-E(G_0)|$. If there is a line e such that e connects V_1 to V_2 but $e\notin E(G_0)$ then we are done. Suppose that no such line exists. Let M be a perfect matching in $G-V(G_0)$. Let e be any line connecting V_1 to $V(G)-V_1-V_2$ (e.g. the first line on a path in $G-E(G_0)$, connecting V_1 to V_2). Let F be a perfect matching in G containing e and P the connected component of $F\cup M$ containing e. Then P is an ear of G_0 starting at V_1.

If P ends at V_2, then we are done. If P ends at V_1, then consider the nice subgraph $G_0'=G_0\cup P$ and the set $V_1'=V_1\cup V(P)$. By the induction hypothesis, G_0' has an ear Q, connecting V_1' to V_2. But then Q, together with the appropriate portion of P, forms an ear of G_0 connecting V_1 to V_2. □

For bipartite graphs we can find ears of nice subgraphs with more restrictive properties.

Lemma 1.9. (Bipartite Ear Selection Lemma). *Let G be a bipartite matching-covered graph with bipartition $\{U, W\}$. Let G_1 and G_2 be vertex-disjoint non-empty subgraphs of G such that $G_1\cup G_2$ is nice and $|V(G_1)\cap U|\geqq|V(G_1)\cap W|$.* Then *$G_1\cup G_2$ has an ear connecting $V(G_1)\cap U$ to $V(G_2)\cap W$.*

Proof. By induction on $|E(G)-E(G_1)-E(G_2)|$. If this is 0 then the assertion is void (the hypothesis is not fulfilled). Since G is matching-covered, it follows that

$$\left|\Gamma\big(V(G_1)\cap U\big)\right| > |V(G_1)\cap U| \geqq |V(G_1)\cap W|,$$

and hence there exists an edge e joining $V(G_1)\cap U$ to $W-V(G_1)$.

Let M be a perfect matching of $G-V(G_1\cup G_2)$, F a perfect matching of G containing e, and consider the connected component P of $F\cup M$ containg e. P is an odd path. with one endpoint in $V(G_1)\cap U$. If the other endpoint of P is in $V(G_2)\cap W$, then we are finished. If the other endpoint of P is in $V(G_1)\cap W$, then consider $G_1'=G_1\cap P$. Since $G_1'\cup G_2$ is nice (in fact $M'=M-E(P)$ is a perfect matching in $G-V(G_1'\cup G_2)$), and $|V(G_1')\cap U|\geqq|V(G_1)\cap W|$, we may apply induction and find an M'-altermating path Q connecting $V(G_1')\cap U$ to $V(G_2)\cap W$. But then Q, together with the appropriate portion of P, forms an M-alternating path connecting $V(G_1)\cap U$ to $V(G_2)\cap W$. □

Let G be a non-bipartite matching covered graph and consider a strong ear-decomposition of G in which every step contains at most two ears. It is an important, but unsolved question to find a strong ear-decomposition with as few 2-ear steps parantheses as possible (note that since the number of ears in any strong ear-decomposition is $|E(G)|-|V(G)|+1$, this is the same problem as finding a strong ear-decomposition with the maximum number of steps.) For ramifications of this problem, see Naddef and Pulleyblank [18]. Here we prove the following, loosely connected result. Let G be a matching-covered graph and G' a nice matching-covered subgraph of G. We say that G' is a *splitting subgraph* if for every connected component T of $G-E(G')$, $V(T)\cap V(G')$ is contained in one class of $\mathscr{P}(G')$. In other words, no path in $G-E(G')$ connects two points in different classes of $\mathscr{P}(G')$. It is clear that if G' is splitting then in any ear-decomposition

$$G = G'\cup P_1\cup \ldots \cup P_k,$$

the subgraph $G'\cup P_1$ is not matching-covered. So if G' occurs in any strong ear-decomposition of G, then the next step must consist of the addition of at least two ears. The following theorem is a certain converse of this observation.

Theorem 1.10. *Every matching-covered graph G has a strong ear-decomposition such that whenever a bracket contains more then one ear, the subgraph constructed up to that point is splitting.*

Roughly speaking, in this ear-decomposition we add two ears only if we are forced by the fact that the subgraph already constructed is splitting.

Proof. Starting with an arbitrary edge e_0, we add one or two ears repeatedly so that we always obtain a matching-covered nice subgraph G' of G and whenever possible, we add only one ear.

Assume that at a certain step we cannot add one ear to the current G' so that the resulting graph is matching-covered. If G' does not split G then there is a path P connecting two distinct classes $S_1, S_2\in\mathscr{P}(G)$. Thus by The Ear Selection Lemma, G' has an ear P connecting S_1 to $V(G')-S_1$. Then $G'+P$ is matching-covered and hence P could be added to G', a contradiction. □

We end this section with quoting the following result of G.H.C. Little on matching-covered graphs, for sake of later reference.

Theorem 1.11. *Any two lines of a matching-covered graph are contained in a nice cycle.* □

2. Non-bipartite matching-covered graphs

We have discussed in the previous section how far can the addition of two ears be postponed. It is another interesting question how soon can we "get over with" the addition of two ears. We can answer this question for the first 2-ear step, by showing that this can always be done in either the second or the third step. This result will have some non-trivial applications.

Theorem 2.1. *Let G be a non-bipartite graph. Then G has a non-refinable ear-decomposition such that either the second or the third step is the addition of 2 ears.*

The result trivially follows from (in fact, it is equivalent to) the following theorem. Let K_n denote the complete n-graph and R_n denote the n-prism, i.e. the cartesian sum of K_2 and an n-cycle. With this notation, K_4 and R_3 are non-bipartite matching-covered graphs with the property that in every ear-decomposition of them, the last step is the only 2-ear addition step (Figure 1).

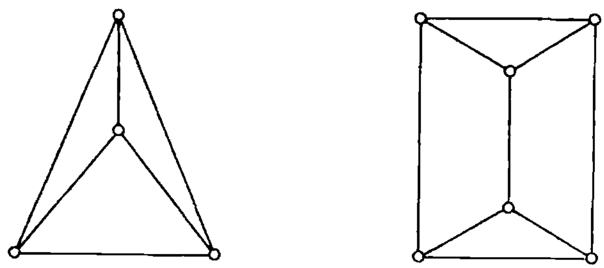

Fig. 1

Theorem 2.2. *Let G be a non-bipartite matching-covered graph all whose nice matching-covered proper subgraphs are bipartite. Then G is isomorphic to an even subdivision of K_4 or R_3.*

Proof. By induction on $|V(G)|$. We may assume that G has no point of degree 2. For suppose that $x\in V(G)$ is adjacent to y and z but to no other point. Identify x, y and z. The resulting graph is trivially non-bipartite and matching-covered, and so, it contains an even subdivision G_1 of K_4 or R_3 as a nice subgraph. Clearly, $G_1+xy+yz$ corresponds to a nice subgraph of the original graph G which contains a nice even subdivision of one of these two graphs. By the minimality hypothesis, G itself must be an even subdivision of K_4 or R_3.

It also follows that G is 3-line-connected. For suppose that G has a pair $\{e,f\}$ of lines such that $G-e-f$ is disconnected. Since G is 2-line-connected, $G-e-f$ has then exactly two connected components G_1 and G_2, and e and f connect G_1 to G_2. Let $e=x_1x_2$, $f=y_1y_2$ where $x_i, y_i\in V(G_i)$.

Case 1. G_1 and G_2 are even. Then one of $G_1+x_1y_1$, $G_2+x_2y_2$, say the first, must be non-bipartite. It is easy to see that $G_1+x_1y_1$ is matching-covered and hence it contains an even subdivision H of K_4 or R_3 as a nice subgraph. If H does not contain x_1y_1 we are done. So suppose that $x_1y_1\in E(H)$. Further, since $G_2+x_2y_2$ is matching-covered and not just K_2, it contains a nice cycle C through x_2y_2. But then $(H-x_1y_1)+(C-x_2y_2)+x_1x_2+y_1y_2$ is an even subdivision of K_4 or R_3 contained in G.
Case 2. G_1 and G_2 are odd. Then let G_1' and G_2' denote the graphs obtained from G by contracting G_2 and G_1 to a single point, respectively. G_1' and G_2' are matching-covered and one of them, G_1' say, is non-bipartite. We conclude just like in Case 1.

Now we turn to the more substantial part of the proof. Consider any non-refinable ear-decomposition of G. Then the graph G' before the last step is bipartite. Since G is non-bipartite, it must arise from G' by attaching two ears. Since G has no point of degree 2, these two ears are of length 1. Let e_1 and e_2 be the two lines forming these last ears. Then it is easily verified that e_1 and e_2 are spanned by different color classes of G' and that every nice cycle (in fact, every even cycle) containing one of them must also contain the other.

Let C be a nice cycle through e_1 (and so through e_2), and consider a non-definable ear-decomposition of G starting with C. As before it follows that G arises from a bipartise G'' by attaching two edges f_1 and f_2, one in each color class.

Let C' a nice cycle of G containing both e_1 and f_1 (this exists by Litte's Theorem 1.11). Then by the above, C' also contains e_2 and f_2.

Consider an ear-decomposition of G starting with C', and let, similarly as before, h_1 and h_2 be the two lines to add to G''' to obtain G. Thus each of the pairs $\{e_1, e_2\}$, $\{f_1, f_2\}$ and $\{h_1, h_2\}$ has the property that deleting its elements from G the remaining graph is bipartite.

Let C be a nice cycle in $G-e_1-e_2$ containing both endpoints of e_1 (such a cycle exists by Little's theorem). The line e_1 forms with the two arcs of C two odd cycles. Since $G-f_1-f_2$ is bipartite, it follows that f_1 and f_2 are lines of C and if E_1 and E_2 are the two components of $C-f_1-f_2$ then e_1 connects E_1 and E_2. Further, E_1 and E_2 are odd paths, as f_1 and f_2 are spanned by different color classes of the bigraph $G-f_1-f_2$.

Applying the Bipartite Ear Selection Lemma, we obtain a path. $P \subset G-f_1-f_2$ alternating with respect to some perfect matching of $G-V(C)$, which connects E_1 to E_2 and whose endpoint on E_1 is at an odd distence from the endpoint of e_1 on E_1. Since $G-f_1-f_2$ is bipartite, the endpoints of e_1 and P on E_2 are also at an odd distance. If the endpoints of e_1 and P separate each other on C, then we have an even subdivision of K_4. So suppose that they do not. Then e_1 and P form, with appropriate arcs of C, two disjoint odd circuits C_1 and C_2 such that $C_1 \cup C_2$ is a nice subgraph. Since $G-e_1-e_2$ is bipartite, it follows that $e_i \in E(C_i)$ and similarly, $f_i, h_i \in E(C_i)$ (if the subscripts are chosen appropriately).

Let E_i, F_i and H_i denote the components of $C_i-\{e_i, f_i, h_i\}$, so that E_i and e_i, F_i and f_i, as well as H_i and h_i are disjoint (Figure 2).

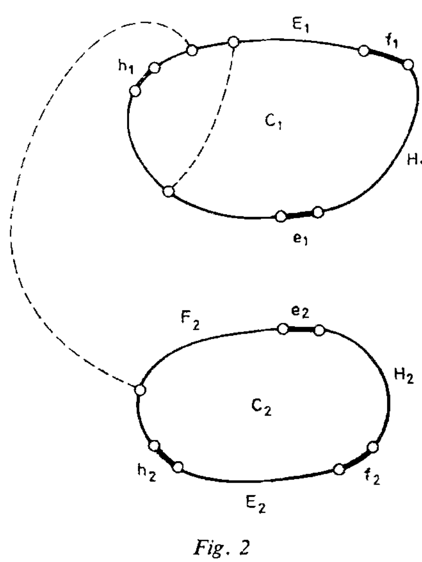

Fig. 2

Claim 1. *No path in* $G-\{e_1, e_2, f_1, f_2, h_1, h_2\}$ *connects two points of* C_i *on different arcs* E_i, F_i, H_i.

Proof. Suppose that there is such a path P connecting a point of E_i to a point of F_i. Then P, together with one of the two arcs of C_i connecting its endpoints, forms an odd cycle C_0. But then C_0 misses at least one of the pairs $\{e_1, e_2\}$, $\{f_1, f_2\}$ and $\{h_1, h_2\}$, which is a contradiction. □

Claim 2. *If a path openly disjoint from* $C_1 \cup C_2$ *connects* C_1 *to* C_2 *then it connects* E_1 *to* E_2 *or* F_1 *to* F_2 *or* H_1 *to* H_2.

Proof. Suppose that P is such a path connecting E_1 to F_2 (say). Since G is 3-line-connected, $G-\{e_1, f_1\}$ contains a path Q connecting H_1 to $C_1 \cup C_2 \cup P - V(H_1)$. Now the endpoint of Q in $C_1 \cup C_2 \cup P - V(H_1)$ cannot be on $P \cup F_2$, by Claim 1. Let R_1 be the arc of C_1 connecting the endpoints of P and Q, cotaining f_1 (but not e_1 and h_1), and let R_2 be the arc of C_2 connecting the endpoints of P and Q such that the cycle $C_0 = R_1 \cup P \cup R_2 \cup Q$ is odd. Then C_0 misses either both e_1 and e_2 or both h_1 and h_2, which is a contradiction. □

Since C_1 is odd, at least one of S_1, F_1, H_1 is of even length; let, say, E_1 be of even length. Let M be a perfect matching in $G-V(C_1)-V(C_2)$.

Claim 3. *There exists an M-alternating path P connecting a point* $x_1 \in V(E_1)$ *to a point* $x_2 \in V(E_2)$; *furthermore,* x_1 *divides* E_1 *into two even paths.*

Proof. To prove this claim, consider the matching-covered bigraph $G-e_1-e_2$, the subgraphs $G_1 = E_1$, $G_2 = E_2 \cup F_1 \cup F_2 \cup H_1 \cup H_2$, and the 2-coloration $\{U, W\}$ of $G-e_1-e_2$ in which the points of $U \cap V(E_1)$ divide E_1 into two even paths. Then the Bipartite Ear Selection Lemma implies that $G-e_1-e_2$ contains an M-alternating path P connecting $U \cap V(E_1)$ to $W \cap (V(E_2) \cup V(F_1) \cup V(F_2) \cup V(H_1) \cup V(H_2))$. By Claims 1 and 2, P must end at $W \cap V(E_2)$. □

By the same argument we find an M-alternating path Q connecting a point $y_1 \in V(F_1)$ to a point $y_2 \in V(F_2)$, where y_1 is at an even distance on F_1 from h_1; and an M-alternating path R connecting a point $z_1 \in V(H_1)$ to a point $z_2 \in V(H_2)$, where z_1 is at an even distance on H_1 from f_1. Thus x_1, y_1 and z_1 divide C_1 into three odd arcs. Claim 1 implies that P, Q and R are vertex-disjoint, and the fact that $G-e_1-e_2$ is bipartite implies that x_2, y_2 and z_2 divide C_2 into three odd arcs. Thus $C_1 \cup C_2 \cup$ $\cup P \cup Q \cup R$ is a nice subgraph and an even subdivision of R_3. □

3. The König property

Let, for a graph G, $\nu(G)$ denote the maximum size of a matching and $\tau(G)$ the minimum number of points covering all lines. Obviously, $\nu(G) \leqq \tau(G)$. We say that G has the König property if $\nu(G) = \tau(G)$. (The theorem of König asserts that every bipartite graph has this property.)

Sterboul [20] and Deming [3] found the following characterization of graphs with the König property. Let M be a maximum matching in G. A *monocle* (with respect to M) is the union of an odd cycle C and an even path P such that P connects a point of C to a point missed by M, has no other point in common with C, and M meets

$C \cup P$ in a maximum matching of $C \cup P$. A *binocle* (with respect to M) is the union of two odd cycles C_1 and C_2 and an odd path P such that P connects a point of C_1 to a point of C_2, has no other point in common with $C_1 \cup C_2$, and $M \cap P$ is a perfect matching of P and $M \cap C_i$ is a maximum matching of C_i. (The two cycles C_1 and C_2 do not have to be vertex-disjoint!)

Theorem 3.1. (Deming [3], Sterboul [20]). *Let G be a graph and M a maximum matching of G. Then G has the König property if and only if it contains no monocle or binocle with respect to M.* □

Here we offer two other characterizations of the König property. This property is by definition in *NP*. Each of the three characterizations imply that it is also in CO-NP. The next characterization easily yields that the König property is in P (this also follows from the work of Sterboul and Deming).

We shall need the following important theorem of Edmonds [4] and Gallai [6]. Let G be any graph. Let $D(G)$ denote the set of points of G which are missed by some maximum matching. Let $A(G)$ denote the set of neighbors of $D(G)$ in $V(G)-D(G)$; and let $C(G)=V(G)-A(G)-D(G)$.

Theorem 3.2. (Edmonds [4], Gallai [6]). *The subgraph spanned by $C(G)$ has a perfect matching. Every connected component of the subgraph spanned by $D(G)$ is critical. The number of components spanned by $D(G)$ is $|A(G)|+|V(G)|-2\nu(G)$. Every maximum matching of G consists of a perfect matching of $C(G)$, a matching of the points of A with points in different components of G, and a maximum matching of each component spanned by $D(G)$.* □

It is also important to point out that Edmonds' matching algorithm [4] ends up with this partition $\{A(G), C(G), D(G)\}$ of $V(G)$.

Using this result, the study of the König property can easily be reduced to the case when the graph in question has a perfect matching.

Lemma 3.3. *A graph G has the König property iff $D(G)$ is an independent set of points and the subgraph induced by $C(G)$ has the König property.*

Proof. Let $x, y \in D(G)$ and assume that x and y are adjacent. Let T be a minimum point-cover of G, then one of x and y, say x, belongs to T. By the definition of $D(G)$, $G-x$ has a matching M of size $\nu(G)$. Since T must cover all edges of M, it follows that $|T| \geqq |M|+1>\nu(G)$. So G does not have the König property.

Forthermore, assume that the subgraph G_1 induced by $C(G)$ does not have the König property, and let again T be a minimum point-cover of G. Then $T \cap C(G)$ is a point-cover of G_1 and hence $|T \cap C(G_1)|>\nu(G_1)=|V(G)|/2$. Further, $|T \cap (A(G) \cup D(G))| \geqq |A(G)|$, since $A(G) \cup D(G)$ spans $|A(G)|$ independent lines by the properties of the Edmonds—Gallai decomposition. Hence $|T|>|A(G)|+\frac{1}{2}|C(G)|=\nu(G)$, and so G fails to have the König property again.

Conversely, assume that G_1 has the König property and $D(G)$ is an independent set. Let T be a minimum point-cover of G_1, then $T \cup A(G)$ is a point-cover of G and

$$|T \cup A(G)| = \tfrac{1}{2}|C(G)|+|A(G)| = \nu(G).$$

Thus G has the König property. □

A *perfect k-matching* in a graph G is an assignment of non-negative integers to the lines in such a way that the sum of the values of lines adjacent to any given point is k. Thus in every perfect 2-matching those lines with value 2 form a matching M, and those lines with value 1 form vertexdisjoint cycles which cover $V(G)-V(M)$. We say that a line *occurs* in a perfect k-matching if its value is positive.

Two perfect matchings give rise to a perfect 2-matching, by letting the value of a line be the number of perfect matchings among these two containing it. Not every perfect 2-matching, however, arises this way: a perfect 2-matching is the sum of two perfect matchings iff every cycle formed by edges of value 1 is even.

Theorem 3.4. *A graph G with a perfect matching has the König property iff those lines which occur in some perfect 2-matching form a bipartite graph.*

Proof. I. Assume first that there exists an odd cycle C all whose edges occur in perfect 2-matchings. Let T be a point-cover, then T must contain two points x, y adjacent on C. Let ω be a perfect 2-matching in which xy occurs. Then

$$2|T| = \sum_{u\in T}\sum_{uv\in E(G)} \omega(uv) \geqq \sum_{uv\in E(G)} \omega(uv)+\omega(xy) \geqq |V(G)|+1,$$

whence

$$|T| > \tfrac{1}{2}|V(G)| = \nu(G).$$

Thus G does not have the König property.

II. Assume that those lines occuring in perfect 2-matchings form a bipartite graph G_0.

Since G has a perfect 2-matching, $|\Gamma(X)|\geqq|X|$ for every independent $X\subseteqq V(G)$.

Assume first that G contains an independent set $A\subseteqq V(G)$ such that $|\Gamma(A)|=$ $=|A|$. Consider the subgraph G_1 induced by $A\cup\Gamma(A)$ and the subgraph $G_2=$ $=G-A-\Gamma(A)$. Obviously, any perfect matching of G consists of a perfect matching of G_1 and one of G_2. Furthermore, every perfect 2-matching of G_2 extends to a perfect 2-matching of G. Hence it follows that those edges of G_2 occuring in perfect 2-matchings form a bipartite graph and so, by induction, we may assume that G_2 contains a point-cover T of size $\frac{1}{2}|V(G_2)|=\frac{1}{2}|V(G)|-|A|$. Then $T\cup\Gamma(A)$ is a point-cover of G of size $\frac{1}{2}|V(G)|$.

Second, assume that for every independent set $A\subseteqq V(G)$ we have $|\Gamma(A)|>$ $>|A|$. We claim that every line of G occurs in a perfect 2-matching. For let $xy\in$ $\in E(G)$. Then every independent subset $A\subseteqq V(G-x)$ has at least $|A|$ neighbors in $G-x$, and so by a theorem of Tutte [22], $G-x$ has a perfect 2-matching ω_1. Similarly $G-y$ has a perfect 2-matching ω_2. Then $\omega_1+\omega_2+2xy$ is a perfect 4-matching of G in which xy occurs. Since every perfect 4-matching is the sum of two perfect 2-matchings by Petersen's theorem, we obtain a perfect 2-matching containing xy.

(Remark: This conclusion would also follow from the results of Berge [2] on regularizable graphs.)

Now it follows that G is bipartite and hence $\nu=\tau$ holds true by König's theorem. □

Next we prove another characterization of graphs with the König property, more closely related to the results of Deming and Sterboul.

Theorem 3.5. *Let G be a connected graph with a perfect matching. Then G has the König property iff it does not contain an even subdivision of the graphs in Figure 3 as a nice subgraph.*

Proof. If G contains either one of the graphs in Figure 3 as a nice subgraph then a simple calculation shows that every point-cover of G has more than $\frac{1}{2}|V(G)|$ points and so G does not have the König property.

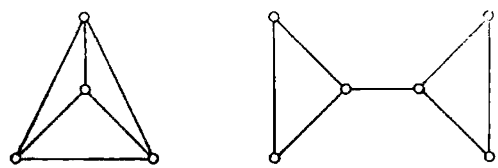

Fig. 3

Assume that G does not have the König property. Then by Theorem 3.4., those lines of G which occur in any perfect 2-matching form a non-bipartite subgraph G_0. Suppose first that no perfect 2-matching of G contains an odd cycle. Then every perfect 2-matching of G is the sum of two perfect matchings and so every line of G_0 occurs in a perfect matching. Hence every connucted component of G_0 is matching-covered. Since G_0 is nonbipartite, it has a component G_1 which is non-bipartite. Thus G_1 contains an even subdivision of one of the two graphs in Figure 1 as a nice subgraph. Hence G contains an even subdivision of one of the two graphs in Figure 3 as a nice subgraph.

Second, suppose that G has a perfect 2-matching ω which contains an odd cycle.

If ω contains $2k>2$ odd cycles, then let G_0 be the union of these. Then G_0 is a nice subgraph of G and since the connected components of G are even, there must exist two odd cycles of G_0 belonging to the same component of G. Hence by the Ear Selection Lemma, there is an ear P of G_0 connecting two odd cycles C_1 and C_2 of G_0. Then $C_1 \cup C_2 \cup P$ has a perfect matching M. Replacing C_1, C_2 and $\omega|P$ by $2M$ in ω, we obtain a perfect 2-matching with $2k-2$ odd cycles. Repeating this procedure if necessary, we obtain a perfect 2-matching ω_0 with exactly 2 odd cycles C_1 and C_2.

As above, it follows that $C_1 \cup C_2$ has an ear P connecting C_1 and C_2. Thus $C_1 \cup C_2 \cup P$ is an even subdivision of the second graph in Figure 3, occuring in G as a nice subgraph. □

As an application of this result, we derive a theorem of Andrásfai [1]. Let G be a graph. We say that G is *τ-critical* if $\tau(G')<\tau(G)$ for each proper subgraph G' of G. Erdős and Gallai [5] proved that, for every τ-critical graph G, $|V(G)|\leq 2\tau(G)$, and that equality holds iff G is a matching. It follows from the results of Hajnal [7] that if G is a connected τ-critical graph with $|V(G)|=2\tau(G)-1$ then G is an odd cycle. Andrásfai [1] proved that if G is a connected τ-critical graph with $|V(G)|= =2\tau(G)-2$ then G is an even subdivision of K_4. Surányi [21] determined those connected τ-critical graphs with $|V(G)|=2\nu(G)-3$ and Lovász [14] proved that for every $\delta>0$ there exist a finite number of graphs such that their even subdivisions are precisely the connected τ-critical graphs with $|V(G)|=2\nu(G)-\delta$.

Let us derive Andrásfai's result from Theorem 3.5. We shall comment on the relationship between matchings and the other results mentioned above in the next section.

So let G be a connected τ-critical graph with $|V(G)|=2\tau(G)-2$. By a theorem of Hajnal, G has a perfect 2-matching ω. The perfect 2-matching ω cannot contain an odd cycle, since then the subgraph formed by the edges occuring in this perfect 2-matching would have $\tau(G')=\tau(G)$, contradicting the hypothesis that G is τ-critical ($G'\neq G$ as G is connected).

So ω contains no odd cycle and hence G has a perfect matching. Since $\tau(G)= =\frac{1}{2}(|V(G)+2)>\nu(G)$, G does not have the König property and so by Theorem 3.5, it contains an even subdivision of one of the two graphs in Figure 2 as a nice subgraph. But G cannot contain an even subdivision of the second, since then we would find a perfect 2-matching with an odd cycle. So G contains an even subdivision G_0 of K_4 as a nice subgraph. But then, if G' is the subgraph consisting of G_0 and a perfect matching of $G-V(G_0)$, we have $\tau(G')=\tau(G)$. Hence $G'=G$. Since G is connected, this is possible only if $G=G_0$. □

4. Concluding remarks

Theorem 2.2. may be rephrased like this: if G is a matching-covered graph an $|\mathcal{P}(G)|>2$ then G has a nice-matching-covered subgraph G' with $|\mathcal{P}(G')|>2$ and with cyclomatic number $\leqq 4$. In this form, it suggests the following conjecture.

Conjecture 1. *There exists a function $f(k)$ with the following property: every matching-covered graph with $|\mathcal{P}(G)|\geqq k$ has a nice matching-covrede subgraph G' with $|\mathcal{P}(G')|\geqq k$ and with cyclomatic number $\leqq f(k)$.*

A similar conjecture arises from Little's theorem:

Conjecture 2. *There exists a function $g(k)$ such that if G is a matching-covered graph and $e_1, \dots, e_k \in E(G)$, then G has a nice matching-covered subgraph containing $e_1, \dots, e_k$ with cyclomatic number $\leqq g(k)$.*

Finally, we formulate a third related conjecture:

Conjecture 3. *There exists a function $h(s, t)$ such that if G is any graph, $X_1, \dots$ $\dots, X_t \subseteq V(G)$, $|X_i|=s$ and $G-X_i$ has a perfect matching for every $1\leqq i\leqq t$, then G has a nice subgraph G' such that $X_1, \dots, X_t \subseteq V(G')$, $G'-X_i$ has a perfect matching for every $1\leqq i\leqq t$, and G' has cyclomatic number $\leqq h(s, t)$.*

We remark that Conjectures 2 and 3 are true for bipartite graphs (Lovász and Plummer [15]; Lovász [14]). In fact, the validity of conjecture 3 for bipartite graphs is the main lemma in the proof of the finite basis theorem for τ-critical graphs mentioned in the previous section.

Conjecture 2 is valid for $k=2$ (this is just Little's theorem). Conjecture 3 is trivial if $t=2$, but the case $t=3$ is already unsettled. In fact, the general case can be reduced to the case $t=3$.

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